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On a Conjecture Concerning Monotonicity of Zeros of Ultraspherical Polynomials*

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Let C_n^λ, $n=0, 1, ..., \lambda > -1/2$ be the ultraspherical (Gegenbauer) polynomials, orthogonal on $(-1, 1)$ with respect to the weight $(1-x^2)^{\lambda-1/2}$. Denote by $\zeta_{n,k}(\lambda)$, $k=1, ..., [n/2]$ the positive zeros of C_n^λ enumerated in decreasing order. The problem of finding the "extremal" function f for which the products $f(\lambda)\,\zeta_{n,k}(\lambda)$ are increasing functions of λ is of recent interest. Ismail, Letessier, and Askey conjectured that $f(\lambda)=(\lambda+1)^{1/2}$ is the function to solve this problem. We prove the conjecture for sufficiently large n and some related results.

1. INTRODUCTION

The monotonic behaviour of the zeros $\{\zeta_{n,k}(\lambda)\}$ of ultraspherical polynomials is a question of interest both from a mathematical and physical point of view because $\zeta_{n,k}(\lambda)$, $k=1, ..., n$ are the positions of equilibrium of n unit charges in $(-1, 1)$ in the field generated by two charges located at -1 and 1 whose common value is $\lambda/2+1/4$ [12, pp. 140–142]. A well known result of Stieltjes [11; 12, Theorem 6.21.1] asserts that for any fixed $n \geqslant 2$ and k, $1 \leqslant k \leqslant [n/2]$ the positive zeros $\zeta_{n,k}(\lambda)$ of C_n^λ are decreasing functions of λ. It is important to know which is the extremal function f forcing $f(\lambda)\,\zeta_{n,k}(\lambda)$ to increase with λ. In order to answer this question we need a formal definition for extremality. Ahmed, Muldoon and Spigler [1, Remark 4] suggested considering the problem in its greatest generality, namely, for any admissible n and k, to seek a function $f_{n,k}(\lambda)$ for which $Z_{n,k}(\lambda) := f_{n,k}(\lambda)\,\zeta_{n,k}(\lambda)$ is increasing function of λ. Natural requirements are that $f_{n,k}$ be positive and differentiable. Since

$$0 \leqslant Z'_{n,k}(\lambda) = f'_{n,k}(\lambda)\,\zeta_{n,k}(\lambda) + f_{n,k}(\lambda)\,\zeta'_{n,k}(\lambda),$$

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$\zeta_{n,k}(\lambda)>0$, $f_{n,k}(\lambda)>0$ and $\zeta'_{n,k}(\lambda)<0$ then

$$f'_{n,k}(\lambda)/f_{n,k}(\lambda) \geqslant -\zeta'_{n,k}(\lambda)/\zeta_{n,k}(\lambda) \qquad \text{for} \quad \lambda > -1/2. \tag{1}$$

Thus we state the following problem:

(P1) For any fixed n and k, $1 \leqslant k \leqslant [n/2]$, determine the function $f_{n,k}$, positive for $\lambda > -1/2$, for which the products $Z_{n,k}$ are increasing functions of λ and $f'_{n,k}(\lambda)/f_{n,k}(\lambda)$ is minimal.

It is easily seen from (1) that P1 is equivalent to the problem of finding $\zeta'_{n,k}(\lambda)/\zeta_{n,k}(\lambda)$. If instead, we seek a function f which depends on n but not on k, then P1 can be reformulated as

(P2) For any integer $n \geqslant 2$, determine the function $f_n(\lambda)$, positive for $\lambda > -1/2$, such that for each k, $1 \leqslant k \leqslant [n/2]$ the products $f_n(\lambda)\,\zeta_{n,k}(\lambda)$ are increasing functions of λ and $f'_n(\lambda)/f_n(\lambda)$ is minimal.

If we are interested in a universal function the problem is

(P3) Which is the function f, positive for $\lambda > -1/2$, such that for any fixed $n \geqslant 2$ and k, $1 \leqslant k \leqslant [n/2]$, the products $f(\lambda)\,\zeta_{n,k}(\lambda)$ are increasing functions of λ and $f'(\lambda)/f(\lambda)$ is minimal?

This latter problem is the precise formulation of the problem posed in the abstract. The notion "extremal" for the above stated problems is equivalent to the minimality of the quotient $f'_{n,k}/f_{n,k}$, f'_n/f_n and f'/f, respectively. Obviously this requirement determines the unknown function up to a constant factor. Moreover, if we find the solutions $f_{n,k}$ of the most general problem P1 then the solutions f_n of P2 and f of P3 could be consequently derived from

$$f'_n(\lambda)/f_n(\lambda) = \max_{1 \leqslant k \leqslant [n/2]} f'_{n,k}(\lambda)/f_{n,k}(\lambda),$$

and

$$f'(\lambda)/f(\lambda) = \sup_{n \geqslant 2} f'_n(\lambda)/f_n(\lambda).$$

Therefore in order to solve these problems we need upper bounds for $-\zeta'_{n,k}/\zeta_{n,k}$.

There have been many attempts to solve P2 and P3. Laforgia [7] proved that each $\lambda\zeta_{n,k}(\lambda)$ is an increasing function of λ for $0 \leqslant \lambda < 1$. Ahmed, Muldoon and Spigler [1] refined a previous result of Spigler [10] to prove that if $n \geqslant 2$ and $f_n(\lambda) = (2\lambda(2n+1) + 2n^2 + 1)^{1/2}$, then each $f_n(\lambda)\,\zeta_{n,k}(\lambda)$ increases with λ for $-1/2 < \lambda < 3/2$. Ismail and Letessier [5] conjectured that $\lambda^{1/2}\zeta_{n,k}(\lambda)$ increases with λ, $\lambda \geqslant 0$, and Askey observed that $f(\lambda) = (\lambda+1)^{1/2}$ could be a solution of P3. Askey's suggestion was formulated as Conjecture 3 in [4]. In what follows we shall refer to this as *Ismail–Letessier–Askey* conjecture (ILAC). Note that if one proves that each $(\lambda+1)^{1/2}\,\zeta_{n,k}(\lambda)$ is an increasing function of λ for $\lambda > -1/2$ then $f(\lambda) = (\lambda+1)^{1/2}$ will be the solution of P3 because $\zeta_{2,1} = (2(\lambda+1))^{-1/2}$. It follows immediately from [1, Eq. (3.7)] that ILAC is true for $-1/2 < \lambda \leqslant 3/2$.

The purpose of this paper is to establish some monotonicity results concerning multiples of positive zeros of C_n^λ. First we prove ILAC for sufficiently large n.

THEOREM 1. *Let v be a nonnegative integer. Then for any $n > 1 + (v^2 + 3v + 3/2)^{1/2}$ and k, $1 \leqslant k \leqslant [n/2]$ the products $(\lambda+1)^{1/2}\,\zeta_{n,k}(k)$ are increasing functions of λ for $-1/2 < \lambda \leqslant 3/2 + v$.*

This result verifies ILAC for an interval with length $v+2$ if $n > v + 5/2$. On the other hand direct calculations show that the conjecture is true if $2 \leqslant n \leqslant 5$. Then one easily gets

COROLLARY 1. *If $n > 2$ and $1 \leqslant k \leqslant [n/2]$ then $(\lambda+1)^{1/2}\,\zeta_{n,k}(\lambda)$ is an increasing function of λ for $-1/2 < \lambda \leqslant 9/2$.*

A result which holds for $\lambda \geqslant 1/2$ follows.

THEOREM 2. *Let $\lambda \geqslant 1/2$. Then for any admissible n and k the function*

$$(\lambda - 1/2)^{1/2}\,\zeta_{n,k}(\lambda) \tag{2}$$

increases as λ increases.

Despite the fact that the function $f(\lambda) = (\lambda - 1/2)^{1/2}$ is not the extremal one, it is a universal function which forces the products (2) to increase on $[1/2, \infty)$.

The next theorem concerns the monotonic behaviour of the largest zero of C_n^λ.

THEOREM 3. *Let $\lambda > -1/2$. Then*

(i) *for any fixed even n*

$$(\lambda+1)^{1/2}\,\zeta_{n,1}(\lambda)$$

is an increasing function of λ.

(ii) *for any fixed odd* $n \geqslant 3$

$$(\lambda+2)^{1/2}\, \zeta_{n,1}(\lambda)$$

is an increasing function of λ.

The function $f(\lambda)=(\lambda+2)^{1/2}$ is the extremal one to set in (ii) because $\zeta_{3,1}=(2(\lambda+2)/3)^{-1/2}$. It seems that Theorem 3 holds not only for the largest zeros of C_n^λ but for all the positive zeros. Thus ILAC could be refined by setting $(\lambda+1)^{1/2}$ for n even and $(\lambda+2)^{1/2}$ for n odd as extremal funtions.

2. Preliminaries

Denote by g_n^λ and h_n^λ the hypergeometric polynomials of degree n

$$g_n^h(x) := {}_2F_1(-n, n+\lambda; 1/2; x)$$

and

$$h_n^\lambda(x) := {}_2F_1(-n, n+\lambda+1; 3/2; x).$$

It is well known that $C_{2n}^\lambda(x^{1/2})$ and $x^{-1/2}C_{2n+1}^\lambda(x^{1/2})$ are constant multiples of $g_n^\lambda(x)$ and $h_n^\lambda(x)$, respectively (see [12, Section 4.7]). This fact and some simple computations show that g_n^λ and h_n^λ, $n=0, 1, ...$ are orthogonal polynomial sequences on $(0, 1)$ with respect to the weight functions $\omega_e(x)=x^{-1/2}(1-x)^{\lambda-1/2}$ and $\omega_o(x)=x^{1/2}(1-x)^{\lambda-1/2}$, respectively. Another simple consequence is that the zeros of g_n^λ are $\zeta_{2n,k}^2(\lambda)$ and those of h_n^λ are $\zeta_{2n+1,k}^2(\lambda)$, $k=1, ..., n$. On using (28) and (29) in [2, p. 103] we obtain the following recurrence relations for g_n^λ and h_n^λ:

$$\begin{aligned}
g_0^\lambda(x) &= 1\\
g_1^\lambda(x) &= 1-2(\lambda+1)\,x\\
-xg_n^\lambda(x) &= \frac{(2n+1)\,(n+\lambda)}{2(2n+\lambda)(2n+\lambda+1)}\, g_{n+1}^\lambda(x)\\
&\quad - \frac{4n^2+4\lambda n+\lambda-1}{2(2n+\lambda-1)(2n+\lambda+1)}\, g_n^\lambda(x)\\
&\quad + \frac{n(2n+2\lambda-1)}{2(2n+\lambda)(2n+\lambda-1)}\, g_{n-1}^\lambda(x),
\end{aligned} \tag{3}$$

$$\begin{aligned}
h_0^\lambda(x) &= 1\\
h_1^\lambda(x) &= 1-2(\lambda+2)\,x/3
\end{aligned}$$

$$
\begin{aligned}
-xh_n^\lambda(x) = {} & \frac{(2n+3)(n+\lambda+1)}{2(2n+\lambda+1)(2n+\lambda+2)}\, h_{n+1}^\lambda(x) \\
& - \frac{4n^2+4(\lambda+1)\,n+3\lambda}{2(2n+\lambda)(2n+\lambda+2)}\, h_n^\lambda(x) \\
& + \frac{n(2n+2\lambda-1)}{2(2n+\lambda)(2n+\lambda+1)}\, h_{n-1}^\lambda(x).
\end{aligned} \tag{4}
$$

Furthermore, $\{g_n^\lambda\}$ and $\{h_n^\lambda\}$ are *sequences of birth and death process polynomials*. This notion is due to Karlin and McGregor [6]. Every *sequence of parametric birth and death process polynomials* $Q_n(x; \tau)$ is defined by

$$
\begin{aligned}
Q_1(x; \tau) &= 1, \\
Q_1(x; \tau) &= [b_0(\tau) + d_0(\tau) - x]/b_0(\tau), \\
-xQ_n(x; \tau) &= b_n(\tau)\, Q_{n+1}(x; \tau) - (b_n(\tau) + d_n(\tau))\, Q_n(x; \tau) + d_n(\tau)\, Q_{n-1}(x; \tau),
\end{aligned}
$$

where $b_n(\tau) > 0$, $d_{n+1}(\tau) > 0$ for $n \geqslant 0$ and $d_0(\tau) \geqslant 0$. The coefficients $b_n(\tau)$ and $d_n(\tau)$ are called *birth rates* and *death rates*, respectively. Ismail [3, 4] pointed out that if the birth rates $b_n(\tau)$ and the death rates $d_n(\tau)$ are increasing functions of τ, then the largest zero of $Q_n(x; \tau)$ is also an increasing function of τ.

We shall prove two simple lemmas.

LEMMA 1. *Let q be a polynomial of degree $n \geqslant 3$ with distinct real zeros. Suppose that q is even (odd) if n is even (odd). Then every positive zero of q' is an increasing function of any positive zero of q.*

Proof. We prove the lemma for even n. The proof for odd n is similar. Assume without loss of generality that q is a monic polynomial. Since it is even and has only real zeros then $q(x) = (x^2 - x_1^2) \cdots (x^2 - x_{n/2}^2)$, $0 < x_{n/2} < \cdots < x_1$. Suppose that $\xi = x_k$ for some k, $1 \leqslant k \leqslant n/2$. Then $q(x) = (x^2 - \xi^2)\, r(x)$. The polynomial q' is odd and by Rolle's theorem it has $n/2 - 1$ positive zeros η_j, $x_{j+1} < \eta_j < x_j$, $j = 1, ..., n/2 - 1$. Let η be one of these zeros, say $\eta = \eta_m$. We claim that η is an increasing function of ξ. Let $q_\varepsilon(x) := (x^2 - (\xi + \varepsilon)^2)\, r(x)$ for some positive ε. It follows immediately from $q'(\eta) = 0$ that

$$
r'(\eta) = -2\eta r(\eta)/(\eta^2 - \xi^2). \tag{5}
$$

On the other hand $q'_\varepsilon(\eta) = 2\eta r(\eta) + (\eta^2 - (\xi+\varepsilon)^2)\, r'(\eta)$. On using (5) we obtain

$$q'_\varepsilon(\eta) = 2\eta((\xi+\varepsilon)^2 - \xi^2)\, r(\eta)/(\eta^2 - \xi^2).$$

Therefore $\operatorname{sign} q'_\varepsilon(\eta) = \operatorname{sign}(r(\eta)/(\eta^2-\xi^2))$. It is not difficult to see that if $\eta > \xi$ then $\operatorname{sign} r(\eta) = (-1)^m = \operatorname{sign} q(\eta)$. Similarly if $\eta < \xi$ then $\operatorname{sign} r(\eta) = (-1)^m = -\operatorname{sign} q(\eta)$. Hence

$$\operatorname{sign} q'_\varepsilon(\eta) = \operatorname{sign} q(\eta) = (-1)^m. \tag{6}$$

Denote by $x_i(\varepsilon)$, $i = 1, ..., n/2$ the positive zeros of q_ε. By Rolle's theorem q'_ε has exactly one zero $\eta(\varepsilon) \in (x_{m+1}(\varepsilon), x_m(\varepsilon))$. On the other hand $\operatorname{sign} q'_\varepsilon(x_{m+1}(\varepsilon)) = (-1)^m$ and $\operatorname{sign} q'_\varepsilon(x_m(\varepsilon)) = (-1)^{m+1}$. This fact and (6) give $\eta(\varepsilon) > \eta$.

Lemma 1 is a slight extension of a classical result of V. Markov [8]. It states that if p and q are polynomials of degree n whose zeros are real and interlace then the zeros of p' and q' interlace in the same way as those of p and q. V. Markov's result is formulated and proved as Lemma 2.7.1 in [9].

The repeated application of Lemma 1 yields

COROLLARY 2. *Let q be a polynomial of degree $n \geqslant 3$ with distinct real zeros. Suppose that q is even (odd) if n is even (odd). Then every positive zero of $q^{(\nu)}$, $0 < \nu < n-1$ is an increasing function of any positive zero of q.*

LEMMA 2. *Let $\omega(x)$ be a positive and continuous weight function on the interval (a, b). Denote by $\{p_n(x; \beta)\}$ the polynomials orthogonal with respect to ω on the interval (a, β), $a < \beta \leqslant b$. Then the zeros of $p_n(x; \beta)$ are increasing functions of β.*

Proof. Let $x_1(\beta), ..., x_n(\beta)$ be the zeros of $p_n(x; \beta)$. They are uniquely implicitly determined by the equations

$$\int_a^\beta \frac{p_n^2(x; \beta)}{x - x_i(\beta)}\, \omega(x)\, dx = 0, \qquad i = 1, ..., n.$$

Differentiating the kth equation with respect to β we obtain

$$-2 \sum_{i \neq k} \int_a^\beta \frac{p_n^2(x; \beta)}{(x - x_i(\beta))(x - x_k(\beta))}\, \omega(x)\, dx\, x'_i(\beta)$$
$$- \int_a^\beta \frac{p_n^2(x; \beta)}{(x - x_k(\beta))^2}\, \omega(x)\, dx\, x'_k(\beta)$$
$$+ \omega(\beta)\, p_n^2(\beta; \beta)/(\beta - x_k(\beta)) = 0.$$

All integrals in the sum above vanish because of the orthogonality. Therefore

$$x_k'(\beta)=\omega(\beta)\, p_n^2(\beta;\beta)(\beta-x_k(\beta))^{-1}\Big/\int_a^\beta \frac{p_n^2(x;\beta)}{(x-x_k(\beta))^2}\,\omega(x)\,dx>0.$$

3. Proofs of the Main Results

Proof of Theorem 1. We use induction with respect to ν. For $\nu=0$ the assertion follows from Theorem 3.1 in [1]. Recall that it states that if $n\geqslant 2$ then

$$(\lambda+(2n^2+1)/(4n+2))^{1/2}\,\zeta_{n,k}(\lambda)$$

increases with λ for $-1/2<\lambda\leqslant 3/2$. Since $(2n^2+1)/(4n+2)\geqslant 1$ for $n\geqslant(3/2)^{1/2}$ then our theorem is true for $\nu=0$. Let ν be any fixed positive integer and assume that the assertion holds for all $\nu'<\nu$. We shall prove it for ν. Since $n\geqslant 1+(\nu^2+3\nu+3/2)^{1/2}>1+((\nu-1)^2+3(\nu-1)+3/2)^{1/2}$ then by the induction hypothesis the functions $(\lambda+1)^{1/2}\,\zeta_{n,k}(\lambda)$ increase with λ for $-1/2<\lambda\leqslant\nu+1/2$. It remains to show that they increase for $\lambda\in(\nu+1/2,\nu+3/2]$. The above mentioned result of Ahmed, Muldoon and Spigler is equivalent to the fact that the positive zeros of $C_N^\mu((\mu+(2N^2+1)/(4N+2))^{-1/2}\,x)$ increase for $\mu\in[-1/2,3/2]$. Substituting $\mu=\lambda-\nu$, $N=n+\nu$ we derive the following statement.

The positive zeros of

$$C_{n+\nu}^{\lambda-\nu}((\lambda-\nu+(2(n+\nu)^2+1)/(4n+4\nu+2))^{-1/2}\,x) \tag{7}$$

increase for $\lambda\in(\nu+1/2,\nu+3/2]$.

On the otherhand [12, Section 4.7] $(C_n^\lambda(x))'=2\lambda C_{n-1}^{\lambda+1}(x)$ and then $(C_{n+\nu}^{\lambda-\nu}(x))^{(\nu)}=c(n,\lambda,\nu)\,C_n^\lambda(x)$ with some nonzero constant $c(n,\lambda,\nu)$. Thus

$$\begin{aligned}&[C_{n+\nu}^{\lambda-\nu}((\lambda-\nu+(2(n+\nu)^2+1)/(4n+4\nu+2))^{-1/2}\,x)]^{(\nu)}\\ &\quad=c_1(n,\lambda,\nu)\,C_n^\lambda((\lambda-\nu+(2(n+\nu)^2+1)/(4n+4\nu+2))^{-1/2}\,x).\end{aligned} \tag{8}$$

Observe that $C_{n+\nu}^{\lambda-\nu}((\lambda-\nu+(2(n+\nu)^2+1)/(4n+4\nu+2))^{-1/2}\,x)$ is an even (odd) polynomial in x if $n+\nu$ is even (odd). Hence it follows from Corollary 2, (7) and (8) that the positive zeros of $C_n^\lambda((\lambda-\nu+(2(n+\nu)^2+1)/(4n+4\nu+2))^{-1/2}\,x)$ increase with λ for $\lambda\in(\nu+1/2,\nu+3/2]$. This means that the products

$$(\lambda-\nu+(2(n+\nu)^2+1)/(4n+4\nu+2))^{1/2}\,\zeta_{n,k}(\lambda)$$

are increasing functions of λ for $\lambda \in (\nu+1/2, \nu+3/2]$. Now the result follows from the fact that the inequality

$$\lambda - \nu + (2(n+\nu)^2 + 1)/(4n + 4\nu + 2) \geqslant \lambda + 1$$

is equivalent to the requirement that $n \geqslant 1 + (\nu^2 + 3\nu + 3/2)^{1/2}$.

Proof of Theorem 2. Note that $(\lambda - 1/2)^{1/2}\, \zeta_{n,k}(\lambda)$ increases if and only if $(\lambda - 1/2)\, \zeta^2_{n,k}(\lambda)$ increases. We prove that $(\lambda - 1/2)\, \zeta^2_{2n,k}(\lambda)$, $k = 1, ..., n/2$, are increasing function of λ on $[1/2, \infty)$. The latter products are the zeros of

$$s^\lambda_n(x) := g^\lambda_n(x/(\lambda - 1/2)).$$

It is easily seen that $\{s^\lambda_n\}$ are orthogonal on $(0, \lambda - 1/2)$ with respect to the weight function $\omega_e(x; \lambda) := x^{-1/2}(\lambda - 1/2 - x)^{\lambda - 1/2}$. Let $1/2 < \lambda_1 < \lambda_2$ and $\{s_n\}$ be the polynomials orthogonal on $(0, \lambda_1 - 1/2)$ with respect to the restriction of $\omega_e(x; \lambda_2)$ to $(0, \lambda_1 - 1/2)$. Denote by $\xi_k(\lambda_i)$, $i = 1, 2$ and ξ_k the kth zero of $s^{\lambda_i}_n$, $i = 1, 2$ and s_n, respectively. It must be proved that $\xi_k(\lambda_1) < \xi_k(\lambda_2)$. In order to this we prove that $\xi_k(\lambda_1) < \xi_k$ and $\xi_k < \xi_k(\lambda_2)$. The first inequality follows immediately from Markov's theorem [12, Theorem 6.12.1] and the fact that

$$\frac{\partial}{\partial \lambda} \ln \omega_e(x; \lambda) = \frac{\partial}{\partial \lambda}[(\lambda - 1/2)\ln(\lambda - 1/2 - x)]$$

$$= \ln(\lambda - 1/2 - x) + (\lambda - 1/2)/(\lambda - 1/2 - x)$$

is an increasing function of x for $x \in (0, \lambda)$. The inequality $\xi_k < \xi_k(\lambda_2)$ is a consequence of Lemma 2.

The proof that $(\lambda - 1/2)\, \zeta^2_{n,k}(\lambda)$ increases as λ increases for odd n is similar.

Proof of Theorem 3. First we consider the case of even n. Taking into account the notes in the proof of Theorem 2 we have to prove that the largest zero of $G^\lambda_n(x) := g^\lambda_n(x/(\lambda+1))$ increases with λ for $\lambda > -1/2$. It follows from (3) that $\{G^\lambda_n\}$ are parameteric birth and death process polynomial defined by

$$G^\lambda_0(x) = 1,$$

$$G^\lambda_1(x) = 1 - 2x,$$

$$-xG^\lambda_n(x) = B_n(\lambda)\, G^\lambda_{n+1}(x) - (B_n(\lambda) + D_n(\lambda))\, G^\lambda_n(x) + D_n(\lambda)\, G^\lambda_{n-1}(x),$$

where the birth and death rates are

$$B_n(\lambda)=(\lambda+1)\,\frac{(2n+1)(n+\lambda)}{2(2n+\lambda)(2n+\lambda+1)},$$

$$D_n(\lambda)=(\lambda+1)\,\frac{n(2n+2\lambda-1)}{2(2n+\lambda)(2n+\lambda-1)}.$$

Recall that by the Perron–Frobenius theorem (see [13], used in [3, 4]) the increase of birth rates and death rates yield the increase of the largest zero of G_n^λ.

$$\begin{aligned}(n+1/2)^{-1}\,&(2n+\lambda)^2\,(2n+\lambda+1)^2\,B_n'(\lambda)\\&=(n+2\lambda+1)(2n+\lambda)(2n+\lambda+1)-(\lambda+1)(n+\lambda)(4n+2\lambda+1)\\&=n(3\lambda^2+2(4n+1)\,\lambda+4n^2+2n+1).\end{aligned}$$

Hence $B_n'(\lambda)>0$ for $\lambda\geqslant 0$. The expression in the brackets can be rewritten in the form $4n^2+2(4\lambda+1)\,n+3\lambda^2+2\lambda+1$. Its discriminant is equal to $4\lambda^2-3$. Therefore the expression is positive for $|\lambda|<\sqrt{3}/2$. Thus $B_n'(\lambda)>0$ for $\lambda>-1/2$.

$$\begin{aligned}2n^{-1}&(2n+\lambda)^2\,(2n+\lambda-1)^2\,D_n'(\lambda)\\&=(2n+\lambda)(2n+\lambda-1)(2n+4\lambda+1)-(\lambda+1)(2n+2\lambda-1)(4n+2\lambda-1)\\&=(2n-1)(3\lambda^2+2(4n-1)\,\lambda+4n^2-2n+1).\end{aligned}$$

Obviously $D_n'(\lambda)>0$ for $\lambda\geqslant 0$. The discriminant of the expression in the brackets, $4n^2+2(4\lambda-1)\,n+3\lambda^2-2\lambda+1$, is again equal to $4\lambda^2-3$ and $D_n'(\lambda)>0$ for $\lambda>-1/2$.

When n is odd we have to prove that the largest zero of $H_n^\lambda(x):=h_n^\lambda(x/(\lambda+2))$ increases with λ for $\lambda>-1/2$. H_n^λ are birth and death process polynomials with birth rates and death rates defined by

$$B_n(\lambda)=(\lambda+2)\,\frac{(2n+3)(n+\lambda+1)}{2(2n+\lambda+1)(2n+\lambda+2)},$$

$$D_n(\lambda)=(\lambda+2)\,\frac{n(2n+2\lambda-1)}{2(2n+\lambda)(2n+\lambda+1)}.$$

Since

$$\begin{aligned}(n+3/2)^{-1}\,&(2n+\lambda+1)^2\,(2n+\lambda+2)^2\,B_n'(\lambda)\\&=n(3(\lambda+1)^2+2(4n+1)(\lambda+1)+4n^2+2n+1)\end{aligned}$$

then $B_n'(\lambda) > 0$ for $\lambda > -1$.

$$2n^{-1}(2n+\lambda)^2\,(2n+\lambda-1)^2\,D_n'(\lambda)$$
$$= (6n-1)\,\lambda^2 + 4(4n^2+1)\,\lambda + 8n^3 + 10n + 2 =: r(\lambda).$$

The binomial r attains minimal value for $\lambda_0 < -1/2$ and is positive for $\lambda \geqslant 0$. One easily gets $r(-1/2) = 2n(4n^2 - 2n + 5) + (6n-1)/4 > 0$ for $n > 0$. Thus $r(\lambda) > 0$ for $\lambda > -1/2$. Therefore $D_n(\lambda)$ increases with λ, $\lambda > -1/2$.

Note added in proof. Part (i) of Theorem 3 was proved in a different way in [14].

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